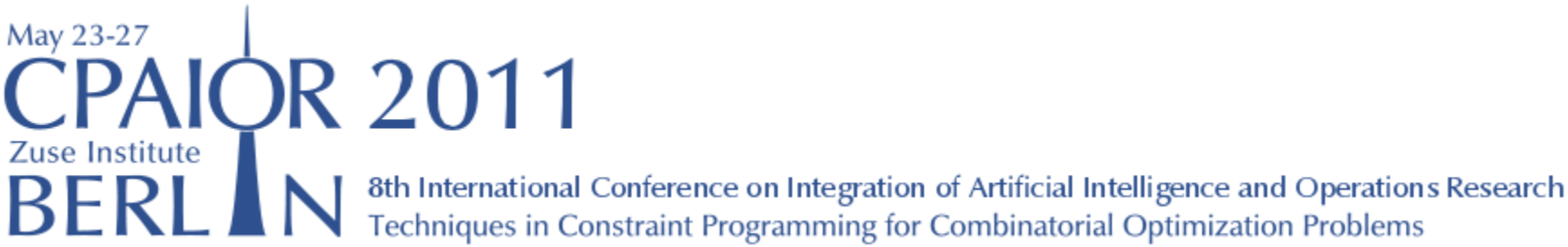
# LP/IP Stories




Martin Grötschel

ZIB, MATHEON and TU Berlin

May 27, 2011

# LP Stories

May 23-27
CPAIOR 2011
Zuse Institute
BERLIN
8th International Conference on Integration of Artificial Intelligence and Operations Research
Techniques in Constraint Programming for Combinatorial Optimization Problems

Martin Grötschel

ZIB, MATHEON and TU Berlin

May 27, 2011

# Remark

Several photos and graphic displays of the original ppt presentation were removed due to unclear copyright status.

# Historical LP Questions (Persons and Time)

1. Who described the first linear equations solver?
2. Who had the first LP solver (without knowing it)?
3. Who formulated the first LP instance?
4. Who described the first LP solver?
5. Who described the first LP solver with impact in practice?
6. Who described the currently most frequently used LP solver?
7. Who implemented the first commercial LP code?
8. Who received a Nobel Prize for LP?
9. Who proved expected polyn. running time of the simplex method first?
10. Who described the first polynomial time linear equations solver?
11. Who had the first polynomial time LP solver (without knowing it)?
12. Who described the first polynomial time LP solver?
13. Who described the first polynomial time LP solver with practical impact?
14. Who described the currently most frequently used barrier LP solver?
15. What is the state of the art in LP solving?

# Who described the first linear equations solver?

- It is impossible to find out who knew what when first.

Just two "references":

- Egyptians and Babylonians considered about 2000 B.C. the solution of special linear equations. But, of course, they described examples and did not describe the methods in "today's style".
- What we call "Gaussian elimination" today has been explicitely described in Chinese "Nine Books of Arithmetic" which is a compendium written in the period 2010 B.C. to A.D. 9, but the methods were probably known long before that.

- Gauss, by the way, never described "Gaussian elimination". He just used it and stated that the linear equations he used can be solved "per eliminationem vulgarem"

# Who had the first LP solver (without knowing it)?

- In 1827 Fourier described a variable elimination method for linear inequalities, today often called Fourier-Motzkin elimination (Motzkin 1936).
- By adding one variable and one inequality, Fourier-Motzkin elimination can be turned into an LP solver.

# Who formulated the first LP instance?

This is, again, unknown, but the usual credit goes to George J. Stigler

| | | |
|---|---|---|
| Min x1 + x2 | | costs |
| 2x1 + x2 | ≥ 3 | protein |
| x1 + 2x2 | ≥ 3 | carbohydrates |
| x1 | ≥ 0 | potatoes |
| x2 | ≥ 0 | beans |

minimizing the cost of food

Stiglers „Diet Problem“:
„The first linear program“

Sets n nutrients / calorie thousands , protein grams , calcium grams , iron milligrams vitamin-a thousand ius, vitamin-b1 milligrams, vitamin-b2 milligrams, niacin milligrams , vitamin-c milligrams /

f foods / wheat , cornmeal , cannedmilk, margarine , cheese , peanut-b , lard liver , porkroast, salmon , greenbeans, cabbage , onions , potatoes spinach, sweet-pot, peaches , prunes , limabeans, navybeans /

Parameter b(n) required daily allowances of nutrients / calorie 3, protein 70 , calcium .8 , iron 12 vitamin-a 5, vitamin-b1 1.8, vitamin-b2 2.7, niacin 18, vitamin-c 75 /

Table a(f,n) nutritive value of foods (per dollar spent)

| | calorie (1000) | protein (g) | calcium (g) | (mg) | iron (1000iu) | vitamin-a (mg) | vitamin-b1 (mg) | vitamin-b2 (mg) | niacin | vitamin-c |
|---|---|---|---|---|---|---|---|---|---|---|
| wheat | 44.7 | 1411 | 2.0 | 365 | 55.4 | 33.3 | 441 | | | |
| cornmeal | | 36 | 897 | 1.7 | 99 | 30.9 | 17.4 | 7.9 | 106 | |
| cannedmilk | | 8.4 | 422 | 15.1 | 9 | 26 | 3 | 23.5 | 11 | 60 |
| margarine | | 20.6 | 17 | .6 | 6 | 55.8 | .2 | | | |
| cheese | 7.4 | 448 | 16.4 | 19 | 28.1 | .8 | 10.3 | 4 | | |
| peanut-b | | 15.7 | 661 | 1 | 48 | | 9.6 | 8.1 | 471 | |
| lard | 41.7 | | | | .2 | | .5 | 5 | | |
| liver | 2.2 | 333 | .2 | 139 | 169.2 | 6.4 | 50.8 | 316 | 525 | |
| porkroast | | 4.4 | 249 | .3 | 37 | | 18.2 | 3.6 | 79 | |
| salmon | 5.8 | 705 | 6.8 | 45 | 3.5 | 1 | 4.9 | 209 | | |
| greenbeans | | 2.4 | 138 | 3.7 | 80 | 69 | 4.3 | 5.8 | 37 | 862 |
| cabbage | | 2.6 | 125 | 4 | 36 | 7.2 | 9 | 4.5 | 26 | 5369 |
| onions | 5.8 | 166 | 3.8 | 59 | 16.6 | 4.7 | 5.9 | 21 | 1184 | |
| potatoes | | 14.3 | 336 | 1.8 | 118 | 6.7 | 29.4 | 7.1 | 198 | 2522 |
| spinach | | 1.1 | 106 | | 138 | 918.4 | 5.7 | 13.8 | 33 | 2755 |
| sweet-pot | | 9.6 | 138 | 2.7 | 54 | 290.7 | 8.4 | 5.4 | 83 | 1912 |
| peaches | | 8.5 | 87 | 1.7 | 173 | 86.8 | 1.2 | 4.3 | 55 | 57 |
| prunes | 12.8 | 99 | 2.5 | 154 | 85.7 | 3.9 | 4.3 | 65 | 257 | |
| limabeans | | 17.4 | 1055 | 3.7 | 459 | 5.1 | 26.9 | 38.2 | 93 | |
| navybeans | | 26.9 | 1691 | 11.4 | 792 | | 38.4 | 24.6 | 217 | |

Positive Variable x(f) dollars of food f to be purchased daily (dollars)

Free Variable cost total food bill (dollars)

Equations nb(n) nutrient balance (units), cb cost balance (dollars) ;

nb(n).. sum(f, a(f,n)*x(f)) =g= b(n); cb.. cost=e= sum(f, x(f));

Model diet stiglers diet problem / nb,cb /;

http://www.gams.com/modlib/libhtml/diet.htm

# Who described the first LP solver?

This is a subtle case.

- 1826/1827 Jean Baptiste Joseph Fourier (1786-1830): rudimentary form of the simplex method in 3 dimensions.
- 1939 L. V. Kantorovitch (1912-1986): Foundations of linear programming (Nobel Prize 1975)
- 1947 G. B. Dantzig (1914-2005): Invention of the (primal) simplex algorithm

# Who described the first LP solver with impact in practice?

1947 G. B. Dantzig (1914-2005):

Invention of the (primal) simplex algorithm

$$\max c^T x$$

$$Ax = b$$

$$x \geq 0$$

# Who described the currently most frequently used LP solver?

- 1954 C.E. Lemke:
  Dual simplex algorithm

- 1953 G.B. Dantzig,
  1954 W. Orchard Hays, and
  1954 G. B. Dantzig & W. Orchard Hays:
  Revised simplex algorithm

# Who implemented the first commercial LP code?

William Orchard-Hayes (in the period 1953-1954)

The first commercial LP-Code was on the market in 1954
(i.e., 57 years ago) and available on an IBM CPC
(card programmable calculator):

Code: Simplex Algorithm with explicit basis inverse, that was recomputed in each step.

Shortly after, Orchard-Hayes implemented a version with product form of the inverse (idea of A. Orden),
Record: 71 variables, 26 constraints, 8 h running time

About 1960: LP became commercially viable, used largely by oil companies.

# Who received a Nobel Prize for LP?

Optimal use of scarce ressources: foundation and economic interpretation of LP

Leonid V. Kantorovich  Tjalling C. Koopmans

Nobel Prize for Economics 1975

# Who proved expected polynomial running time of the simplex method first?

- Borgwardt (1977) Schatteneckenalgorithmus
- Borgwardt (1982)

# Who described the first polynomial time linear equations solver?

- Gaussian elimination (and other linear equations solvers) may lead, if not implemented with care, to exponential size rational numbers!
- Edmonds (1967) showed how to compute "with care".

# Who had the first polynomial time LP solver (without knowing it)?

Fiacco & McCormick (1968)

A.V. Fiacco, G.P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, John Willey & Sons, 1968.

Reprinted as *Classics* in Applied Mathematics 4, *SIAM*, 1990.

# Who described the first polynomial time LP solver?

Доклады Академии наук СССР
1979. Том 244, № 5

УДК 519.95

*МАТЕМАТИКА*

Л. Г. ХАЧИЯН

## ПОЛИНОМИАЛЬНЫЙ АЛГОРИТМ В ЛИНЕЙНОМ ПРОГРАММИРОВАНИИ

*(Представлено академиком А. А. Дородницыным 4 X 1978)*

Рассмотрим систему из $m \geqslant 2$ линейных неравенств относительно $n \geqslant 2$ вещественных переменных $x_1, \ldots, x_j, \ldots, x_n$

$$a_{i1}x_1 + \ldots + a_{in}x_n \leqslant b_i, \quad i = 1, 2, \ldots, m, \tag{1}$$

с целыми коэффициентами $a_{ij}$, $b_i$. Пусть

$$L = \left[ \sum_{i,j=1}^{m,n} \log_2(|a_{ij}|+1) + \sum_{i=1}^{m} \log_2(|b_i|+1) + \log_2 nm \right] + 1 \tag{2}$$

есть длина входа системы, т. е. число символов 0 и 1, необходимых для записи (1) в двоичной системе счисления.

Mathematical Programming Study 14 (1981) 61–68.
North-Holland Publishing Company

# KHACHIYAN'S ALGORITHM FOR LINEAR PROGRAMMING*

Peter GÁCS and Laszlo LOVÁSZ

*Computer Science Department, Stanford University, Stanford, CA 94305, U.S.A.*

Received 10 October 1979

L.G. Khachiyan's algorithm to check the solvability of a system of linear inequalities with integral coefficients is described. The running time of the algorithm is polynomial in the number of digits of the coefficients. It can be applied to solve linear programs in polynomial time.

*Key Words*: Linear Programming, Inequalities, Complexity, Polynomial Algorithms.

## 0. Introduction

L.G. Khachiyan [1, cf. also 2, 3] published a polynomial-bounded algorithm to solve linear programming. These are some notes on this paper. We have ignored his considerations which concern the precision of real computations in order to make the underlying idea clearer; on the other hand, proofs which are missing from his paper are given in Section 2. Let

$$a_i x < b_i \quad (i = 1, \dots, m,\ a_i \in Z^n,\ b_i \in Z) \tag{1}$$

be a system of *strict* linear inequalities with integral coefficients. We present an algorithm which decides whether or not (1) is solvable, and yields a solution if it is. Define

$$L = \sum_{i,j} \log(|a_{ij}| + 1) + \sum_i \log(|b_i| + 1) + \log nm + 1.$$

$L$ is a lower bound on the space needed to state the problem.

# Who described the first polynomial time LP solver with practical impact?

- N. Karmarkar,
  *A new polynomial-time algorithm for linear programming*.
  Combinatorica 4 (1984), no. 4, 373--395.

**United States Patent** [19]

**Karmarkar**

[11] **Patent Number: 4,744,028**

[45] **Date of Patent: May 10, 1988**

[54] **METHODS AND APPARATUS FOR EFFICIENT RESOURCE ALLOCATION**

[75] Inventor: **Narendra K. Karmarkar, Somerset, N.J.**

[73] Assignee: **American Telephone and Telegraph Company, AT&T Bell Laboratories, Murray Hill, N.J.**

[21] Appl. No.: **725,342**

[22] Filed: **Apr. 19, 1985**

[51] **Int. Cl.**[4] ..... **G06F 15/20**; H04Q 3/66; H04M 7/00

[52] **U.S. Cl.** ..... **364/402**

[58] **Field of Search** ..... 364/402; 379/113, 221; 340/524

[56] **References Cited**

U.S. PATENT DOCUMENTS

| | | | |
|---|---|---|---|
| 4,364,115 | 12/1982 | Asai | 364/765 |
| 4,479,176 | 10/1984 | Grimshaw | 369/168 |
| 4,481,600 | 11/1984 | Asai | 364/765 |

# *Breakthrough in Problem Solving*

By JAMES GLEICK

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

### Faster Solutions Seen

These problems are fiendishly complicated systems, often with thousands of variables. They arise in a variety of commercial and government applications, ranging from allocating time on a communications satellite to routing millions of telephone calls over long distances, or whenever a limited, expensive resource must be spread most efficiently among competing users. And investment companies use them in creating portfolios with the best mix of stocks and bonds.

The Bell Labs mathematician, Dr. Narendra Karmarkar, has devised a radically new procedure that may speed the routine handling of such problems by businesses and Government agencies and also make it possible to tackle problems that are now far out of reach.

"This is a path-breaking result," said Dr. Ronald L. Graham, director of mathematical sciences for Bell Labs in Murray Hill, N.J.

"Science has its moments of great progress, and this may well be one of them."

Because problems in linear programming can have billions or more possible answers, even high-speed computers cannot check every one. So computers must use a special procedure, an algorithm, to examine as few answers as possible before finding the best one — typically the one that minimizes cost or maximizes efficiency.

A procedure devised in 1947, the simplex method, is now used for such problems,

Continued on Page A19, Column 1

THE NEW YORK TIMES, November 19, 1984

Karmarkar at Bell Labs: an equation to find a new way through the maze

# Folding the Perfect Corner

*A young Bell scientist makes a major math breakthrough*

Every day 1,200 American Airlines jets crisscross the U.S., Mexico, Canada and the Caribbean, stopping in 110 cities and bearing over 80,000 passengers. More than 4,000 pilots, copilots, flight personnel, maintenance workers and baggage carriers are shuffled among the flights; a total of 3.6 million gal. of high-octane fuel is burned. Nuts, bolts, altimeters, landing gears and the like must be checked at each destination. And while performing these scheduling gymnastics, the company must keep a close eye on costs, projected revenue and profits.

Like American Airlines, thousands of companies must routinely untangle the myriad variables that complicate the efficient distribution of their resources. Solving such monstrous problems requires the use of an abstruse branch of mathematics known as linear programming. It is the kind of math that has frustrated theoreticians for years, and even the fastest and most powerful computers have had great difficulty juggling the bits and pieces of data. Now Narendra Karmarkar, a 28-year-old Indian-born mathematician at Bell Laboratories in Murray Hill, N.J., after only a years' work has cracked the puzzle of linear programming by devising a new algorithm, a step-by-step mathematical formula. He has translated the procedure into a program that should allow computers to track a greater combination of tasks than ever before and in a fraction of the time.

Unlike most advances in theoretical mathematics, Karmarkar's work will have an immediate and major impact on the real world. "Breakthrough is one of the most abused words in science," says Ronald Graham, director of mathematical sciences at Bell Labs. "But this is one situation where it is truly appropriate."

Before the Karmarkar method, linear equations could be solved only in a cumbersome fashion, ironically known as the simplex method, devised by Mathematician George Dantzig in 1947. Problems are conceived of as giant geodesic domes with thousands of sides. Each corner of a facet on the dome

TIME MAGAZINE, December 3, 1984

# Who described the currently most frequently used barrier (interior point) LP solver?

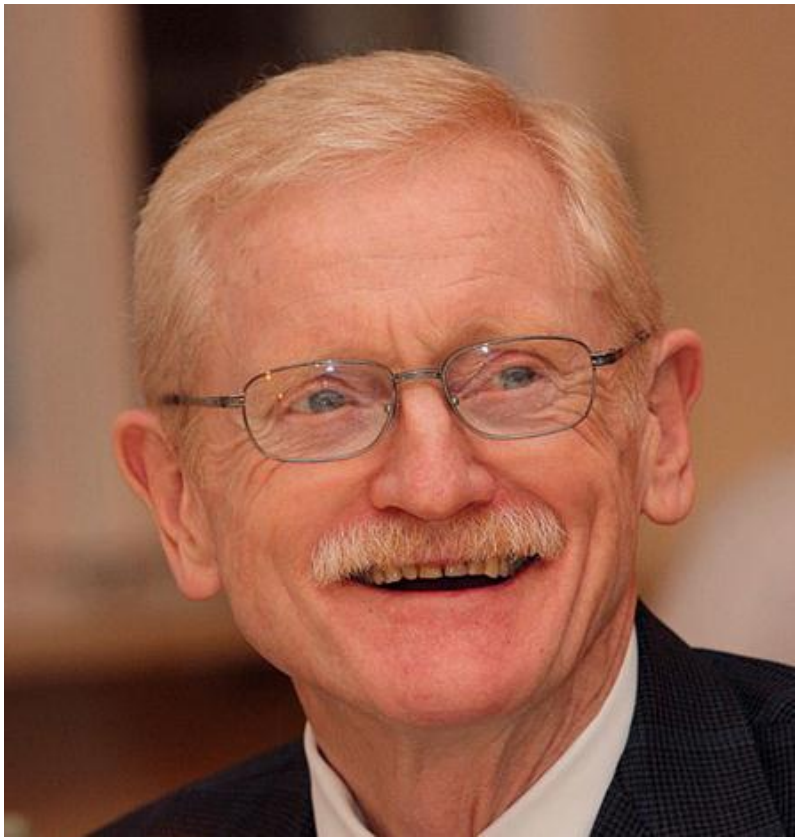
- Masakazu Kojima (1989): primal-dual interior point algorithm

# What is the state of the art in LP solving?

Robert E. Bixby, Solving Real-World Linear Programs: A Decade and More of Progress.
*Operations Research* 50 (2002)3-15.

updated 2004

Bob on September 27, 2010
at his 65th birthday party



# Progress in LP: 1988—2004

Operations Research, Jan 2002, pp. 3—15, updated in 2004

Algorithms (*machine independent*):

| | |
|---|---|
| Primal *versus* best of Primal/Dual/Barrier | 3,300x |
| Machines (workstations →PCs): | 1,600x |
| NET: Algorithm × Machine | 5,300,000x |

(2 months/5300000 ~= 1 second)

Courtesy Bob Bixby

# The latest computational study: Ed Rothberg (Gurobi)

Rothberg slides

LP state of the art - according to Gurobi:
as of September 28, 2010 (Bixby's 65th birthday conference in Erlangen, Germany)

All software producer do computational studies permanently but rarely make them publicly available.

# IP Stories

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# Historical IP Questions (Persons and Time)

1. Which is the most important paper on integer programming?
2. Which is the most important paper on combinatorial algorithms?
3. Who described the first cutting plane algorithm?
4. Who described the first branch&bound algorithm?
5. Who described the first branch&cut algorithm?
6. Who described the first column generation algorithm?
7. Who implemented the first commercial IP code?
8. Who described the first polynomial time IP solver?
9. What is the state of the art in MIP solving?

# Which is the most important paper on integer programming?

- G.B. Dantzig, D.R. Fulkerson, and S. Johnson (1954), *Solution of a Large Scale Traveling Salesman Problem*, P-510, The Rand Corporation.
- G.B. Dantzig, D.R. Fulkerson, and S. Johnson (1954). *Solution of a Large Scale Traveling Salesman Problem*, Operations Research 2, 393-410.

- Martin Grötschel, George L. Nemhauser, *George Dantzig's contributions to integer programming*. Discrete Optimization 5(2): 168-173 (2008)

# Which is the most important paper on combinatorial algorithms?

- H.W. Kuhn, *The Hungarian Method for the assignment problem*, Naval Research Logistic Quarterly, 2 (1955) 83-97.
- Was voted in 2004 as most important paper that ever appeared in Naval Research Logistic Quarterly.
- András Frank, *On Kuhn's Hungarian Method – A tribute from Hungary,*
  Naval Research Logistic Quarterly, 22 (2005) 2-5.

E-mail from Harold Kuhn of March 9, 2006:

Dear Friends:

As participants in the 50th Birthday celebration of the Hungarian Method, you should be among the first to know that Jacobi discovered an algorithm that includes both Koenig's Theorem and the Egervary step. I was told about Jacobi's paper by Francois Ollivier who has a website with the original papers and French and English translations. They were published in Latin after his death and so the work was done prior to 1851!!!

# Who described the first cutting plane algorithm?

- Gomory, R. E. (1958), *Outline of an algorithm for integer solutions to linear programs*,
  Bull. AMS 64, 275 – 278.

- Not the first cutting plane algorithm, though, but the first one with a convergence proof.

# Who described the first branch&bound algorithm?

- A. H. Land, A. G. Doig, *An automatic method of solving discrete programming problems*.
  Econometrica 28, 1960, S. 497–520
- R. J. Dakin, *A tree-search algorithm for mixed integer programming problems*.
  The Computer Journal, Volume 8, 1965, S. 250–255
- J. D. C. Little, K. G. Murty, D. W. Sweeney, C. Karel, *An algorithm for the traveling salesman problem*. Operations Research 11, 1963, 972–989.
- and many predecessors, such as
  W. L. Eastman, *Linear Programming with Pattern Constraints*. Ph.D. Thesis. Department of Economics, Harvard University, Cambridge, Massachusetts, USA, 1958.

# Who described the first branch&cut algorithm?

- Grötschel, Jünger, Reinelt (1984, 1985, 1987)
- Padberg, Rinaldi (1991)

- Branch&Cut&Price

# Who described the first column generation algorithm?

- L. R. Ford, Jr. and D. R. Fulkerson, *A Suggested Computation for Maximal Multi-Commodity Network Flows*
  Management Science, Vol. 5, No. 1, 1958, pp. 97-101
- A. Charnes and M. H. Miller (1956)

# Who implemented the first commercial IP code?

???

MPSX? UMPIRE?

NIC 10424 Karl Kelley
NWG/RFC 345 University of Illinois
May 26, 1972

INTEREST IN MIXED INTEGER PROGRAMMING (MPSX ON 360/91 AT CCN)

MPSX is a newer version of the IBM project MPS, used for integer programming. From what I've been told, MPSX outperforms the previous package. In addition, it has available a feature of mixed integer programming.

# Who implemented the first commercial IP code?

From: Martin Groetschel [mailto:groetschel@zib.de]

Sent: Wednesday, May 25, 2011 5:42 PM

To: Robert E. Bixby

Subject: Question

Bob, who had/developed the first commercial IP and/or MIP code?

Martin

Von: Robert Bixby [mailto:bixby@gurobi.com]
Gesendet: Donnerstag, 26. Mai 2011 00:52
An: 'Martin Groetschel'
Betreff: RE: Question

Boy, I don't think I know the answer to that one. For LP I could probably find it, but I don't recall seeing statements about that for IP/MIP.
My suggestion is to ask Alan Hoffman. He certainly knew about early LP codes and corrected some of my misinformation.

Bob

# Who described the first polynomial time IP solver?

- Well, IP is NP hard!

But

- H.E. Scarf, Production sets with indivisibilities—
  Part I: Generalities, Econometrica 49 (1981) 1–32.

  Part II: The case of two activities, Econometrica 49 (1981) 395-423.
- H. W. Lenstra. *Integer programming with a fixed number of variables.*
  Mathematics of Operations Research, 8(4):538 – 548, 1983.

# What is the state of the art in MIP solving?

THORSTEN KOCH, TOBIAS ACHTERBERG[1],
ERLING ANDERSEN[2], OLIVER BASTERT[3],
TIMO BERTHOLD*, ROBERT E. BIXBY[4],
EMILIE DANNA[5], GERALD GAMRATH,
AMBROS M. GLEIXNER, STEFAN HEINZ*,
ANDREA LODI[6], HANS MITTELMANN[7],
TED RALPHS[8], DOMENICO SALVAGNIN[9],
DANIEL E. STEFFY, KATI WOLTER**

ZIB-Report 10-31

**MIPLIB 2010**

**Mixed Integer Programming Library version 5**

To appear in next issue of Mathematical Programming C (Computation)

# LP/IP Stories

Thanks for your attention

May 27, 2011